Math 2010 Week 10

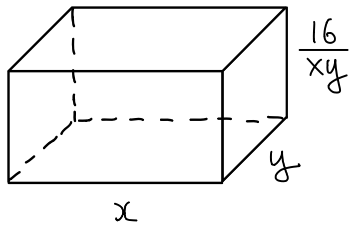
Another example of extrema on unbounded region

eg Make a box (without top) with volume $= 16$

Cost:

Base \$2 / unit area

Side \$0.5 / unit area



Q How to minimize cost?

Sol Want to minimize

$$C(x,y) = 2xy + \left(\frac{16}{xy}x + \frac{16}{xy}y\right)(2)(0.5)$$

$$= 2xy + \frac{16}{x} + \frac{16}{y}$$

on the domain $\Omega = \{(x,y) \in \mathbb{R}^2 : x, y > 0\}$

- Ω is neither closed nor bounded.

  $\therefore$ EVT cannot be applied directly

- C is large if x or y is small or large.

> Strategy: Find a rectange R s.t.
> $C >$ min. of $C|_R$ on ∂R and outside R.

Step 1 Find critical points

$\nabla C = \left(2y - \frac{16}{x^2}, 2x - \frac{16}{y^2}\right)$ exists everywhere

$$\nabla C = \vec{0} \iff \begin{cases} 2y - \frac{16}{x^2} = 0 \\ 2x - \frac{16}{y^2} = 0 \end{cases}$$

$\therefore y = \frac{8}{x^2}, \quad x = \frac{8}{y^2} = \frac{8}{\frac{64}{x^4}} = \frac{x^4}{8}, \quad x > 0$

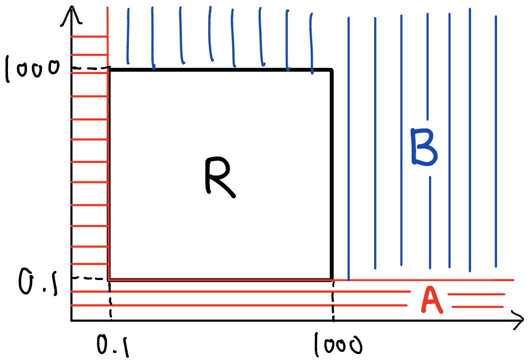
$\Rightarrow x^3 = 8, \ x = 2, \ y = 2$

$\therefore$ Only one critical point $(2,2)$, $C(2,2) = 24$

Step 2 Choose R s.t. $C > 24$ on ∂R and outside R.

$$C(x,y) = 2xy + \frac{16}{x} + \frac{16}{y}$$

One possible choice: $R = [0.1, 1000] \times [0.1, 1000]$



Ⓐ If $x \le 0.1$ or $y \le 0.1$

then $C > \frac{16}{x} + \frac{16}{y} > \frac{16}{0.1} = 160 > 24$

Ⓑ If $(x \ge 0.1, y \ge 1000)$ or $(y \ge 0.1, x \ge 1000)$,

then $C > 2(0.1)(1000) = 200 > 24$

Step 3 Analysis

- R is closed and bounded, C is continuous

  By EVT, $C|_R$ has minimum

- C has only one critical point $(2,2) \in \Omega$

  $(2,2) \in \mathrm{int}(R)$, $C(2,2) = 24$

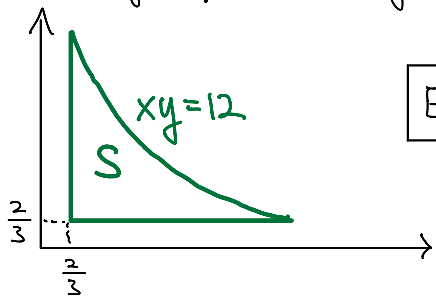
  $C > 24$ on ∂R

  $\Rightarrow C|_R$ has min value 24 at $(2,2)$

- $C > 24$ outside R

$\Rightarrow C$ has min value 24 at $(2,2)$ on Ω

Rmk One may replace R by S below.



Exercise

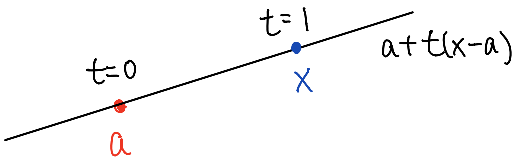
Taylor Series Expansion (Thomas 14.9)

Recall Taylor expansion for 1-variable function $g(t)$ at $t=0$ up to order k

$$\boxed{g(t) = g(0) + g'(0)t + \frac{1}{2!}g''(0)t^2 + \cdots + \frac{1}{k!}g^{(k)}(0)t^k + \text{remainder} \quad (*)}$$

We want a similar formula for a multi-variable function $f(x)$ defined near a, where $x=(x_1,\cdots,x_n)$ $a=(a_1,\cdots,a_n)$

Let $g(t) = f(a+t(x-a))$



If $\|x-a\|$ is small, then for $|t|\le 1$,

$\|t(x-a)\| = |t|\,\|x-a\| \le \|x-a\|$ is small

and $g(t)$ is defined

By $(*)$,

$$f(a+t(x-a)) = g(0) + g'(0)t + \frac{1}{2!}g''(0)t^2 + \cdots + \frac{1}{k!}g^{(k)}(0)t^k + \text{remainder}$$

Put $t=1$,

$$f(x) = g(0) + g'(0) + \frac{1}{2!}g''(0) + \cdots + \frac{1}{k!}g^{(k)}(0) + \text{remainder}$$

Next, express $g^{(k)}(0)$ in terms of f:

$$g(0) = f(a+t(x-a)) = f(a)$$

$$g'(t) = \nabla f(a+t(x-a)) \cdot \frac{d}{dt}\left(a+t(x-a)\right)$$

$$= \nabla f(a+t(x-a)) \cdot (x-a)$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a+t(x-a))\,(x_i-a_i)$$

$$\Rightarrow g'(0) = \nabla f(a)\cdot(x-a)$$

$$= \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i-a_i)$$

$$g''(t) = \frac{d}{dt} g'(t)$$

$$= \sum_{i=1}^{n} \frac{d}{dt}\left[\frac{\partial f}{\partial x_i}(a+t(x-a))\,(x_i-a_i)\right]$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j \partial x_i}(a+t(x-a))(x_j-a_j)(x_i-a_i)$$

$$g''(0) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{\partial^2 f}{\partial x_j\,\partial x_i}(a)\,(x_j-a_j)(x_i-a_i)$$

$\therefore$ Taylor Expansion at a up to order 2 is

$$\boxed{f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i-a_i) + \frac{1}{2!}\sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)\,(x_i-a_i)(x_j-a_j) + \text{remainder}}$$

<u>eg</u> If $n=2$, i.e. $f=f(x,y)$, $a=(x_0,y_0)$

f is C^2 (so $f_{xy}=f_{yx}$), then

$$f(x,y) = f(x_0,y_0) + f_x(x_0,y_0)(x-x_0) + f_y(x_0,y_0)(y-y_0)$$
$$+ \frac{1}{2}\left[f_{xx}(x_0,y_0)(x-x_0)^2 + 2f_{xy}(x_0,y_0)(x-x_0)(y-y_0) + f_{yy}(x_0,y_0)(y-y_0)^2\right]$$
$$+ \text{remainder.}$$

Similarly, the general term is

$$g^{(k)}(0) = \sum_{i_1,\cdots,i_k=1}^{n} \frac{\partial^k f}{\partial x_{i_1}\cdots\partial x_{i_k}}(a)\,(x_{i_1}-a_{i_1})\cdots(x_{i_k}-a_{i_k})$$

Thm (Taylor Theorem)

Let $\Omega \subseteq \mathbb{R}^n$ be open, $f: \Omega \to \mathbb{R}$ be C^k.

Then for any $x, a \in \Omega$,

$$f(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + \frac{1}{2!} \sum_{i,j=1}^{n} \frac{\partial^2 f}{\partial x_i \partial x_j}(a)(x_i - a_i)(x_j - a_j) + \cdots$$

$$+ \frac{1}{k!} \sum_{i_1, \cdots, i_k = 1}^{n} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}(a)(x_{i_1} - a_{i_1}) \cdots (x_{i_k} - a_{i_k}) + \varepsilon_k(x, a)$$

with $\displaystyle \lim_{x \to a} \frac{\varepsilon_k(x,a)}{\|x - a\|^k} = 0$

Defn

$$p_k(x) = f(a) + \sum_{i=1}^{n} \frac{\partial f}{\partial x_i}(a)(x_i - a_i) + \cdots$$

$$+ \frac{1}{k!} \sum_{i_1, \cdots, i_k = 1}^{n} \frac{\partial^k f}{\partial x_{i_1} \cdots \partial x_{i_k}}(a)(x_{i_1} - a_{i_1}) \cdots (x_{i_k} - a_{i_k})$$

is called the <u>k-th order Taylor polynomial</u> of f at a

Rmk

1. $p_1(x) = L(x)$
   = Linearization of f at a

2. p_k and f have equal partial derivatives up to order k at a

eg $f(x,y)=e^x\cos y$

Find 2^{nd} order Taylor polynomial at $a=(0,0)$

Sol
$f_x = e^x \cos y \qquad f_y = -e^x \sin y$

$f_{xx} = e^x \cos y \qquad f_{yx} = -e^x \sin y$

$f_{xy} = -e^x \sin y \qquad f_{yy} = -e^x \cos y$

$\Rightarrow \quad f(0,0)=1,$

$f_x(0,0)=1, \qquad f_y(0,0)=0$

$f_{xx}(0,0)=1 \qquad f_{yy}(0,0)=-1$

$f_{xy}(0,0)=f_{yx}(0,0)=0$

$$P_2(x,y) = f(0,0)+f_x(0,0)x+f_y(0,0)y + \frac{1}{2!}\left(f_{xx}(0,0)x^2+2f_{xy}(0,0)xy+f_{yy}(0,0)y^2\right)$$

$$= 1+x+\frac{1}{2}x^2-\frac{1}{2}y^2$$

How about $P_3(x,y)$ at $(0,0)$?

$$P_3(x,y) = P_2(x,y) + \underbrace{\frac{1}{3!}g^{(3)}(0)}_{3^{rd}\text{ order terms}}$$

$f_{xxx} = e^x \cos y$

$f_{xxy} = f_{xyx} = f_{yxx} = -e^x \sin y$

$f_{xyy} = f_{yxy} = f_{yyx} = -e^x \cos y$

$f_{yyy} = e^x \sin y$

$\Rightarrow \quad f_{xxx}(0,0)=1 \qquad f_{xxy}(0,0)=0$

$f_{xyy}(0,0)=-1 \qquad f_{yyy}(0,0)=0$

$$g^{(3)}(0) = f_{xxx}(0,0)x^3 + 3f_{xxy}(0,0)x^2y + 3f_{xyy}(0,0)xy^2 + f_{yyy}(0,0)y^3$$

$$= x^3-3xy^2$$

$$P_3(x,y) = P_2(x,y)+\frac{1}{3!}(x^3-3xy^2)$$

$$=1+x+\frac{1}{2}x^2-\frac{1}{2}y^2+\frac{1}{6}x^3-\frac{1}{2}xy^2$$

Q If $f = f(x,y,z)$ is C^6, then

Coefficient of xy^2z^3 in $P_6(x,y,z)$ at $(0,0,0)$

is $\alpha f_{xyyzzz}(0,0,0)$. $\alpha = ?$

Rmk A General Taylor's formula for $f(x,y)$ is given on P822 of Thomas' Calculus.

Matrix form for 2nd order Taylor Polynomial

Defn Let $\Omega \subseteq \mathbb{R}^n$ be open, $f: \Omega \to \mathbb{R}$ be C^2.
Then the Hessian matrix of f at $a \in \Omega$ is

$$Hf(a) = \begin{bmatrix} f_{x_1x_1}(a) & \cdots & f_{x_1x_n}(a) \\ \vdots & \ddots & \vdots \\ f_{x_nx_1}(a) & \cdots & f_{x_nx_n}(a) \end{bmatrix}$$

Rmk

① $Hf(a)$ is a symmetric $n\times n$ matrix by the mixed derivatives theorem.

② In Thomas' Calculus, Hessian of f is defined to be the determinant of our Hessian matrix

With the Hessian matrix, the 2nd order Taylor polynomial of f at a can be written as

$$\underset{1\times 1}{P_2(x)} = \underset{1\times 1}{f(a)} + \underset{1\times n}{\nabla f(a)}\underset{n\times 1}{(x-a)} + \frac{1}{2}\underset{1\times n}{(x-a)^T}\underset{n\times n}{Hf(a)}\underset{n\times 1}{(x-a)}$$

where $x, a \in \mathbb{R}^n$ are written as column vectors

$$(x-a)^T = \text{Transpose of } x-a$$
$$= [x_1 - a_1, \cdots, x_n - a_n]$$

__Rmk__

$$(x-a)^T Hf(a)(x-a) = [x_1-a_1, \cdots, x_n-a_n] \begin{bmatrix} f_{x_1x_1}(a) & \cdots & f_{x_1x_n}(a) \\ \vdots & \ddots & \vdots \\ f_{x_nx_1}(a) & \cdots & f_{x_nx_n}(a) \end{bmatrix} \begin{bmatrix} x_1-a_1 \\ \vdots \\ x_n-a_n \end{bmatrix}$$

$$= [x_1-a_1, \cdots, x_n-a_n] \begin{bmatrix} f_{x_1x_1}(a)(x_1-a_1) + \cdots + f_{x_1x_n}(a)(x_n-a_n) \\ \vdots \\ f_{x_nx_1}(a)(x_1-a_1) + \cdots + f_{x_nx_n}(a)(x_n-a_n) \end{bmatrix}$$

$$\begin{aligned} = \; & f_{x_1x_1}(a)(x_1-a_1)(x_1-a_1) + \cdots + f_{x_1x_n}(a)(x_1-a_1)(x_n-a_n) \\ & + \cdots \\ & \vdots \\ & + f_{x_nx_1}(a)(x_1-a_1)(x_n-a_n) + \cdots + f_{x_nx_n}(a)(x_n-a_n)(x_n-a_n) \end{aligned}$$

$$= \sum_{i,j=1}^{n} f_{x_ix_j}(a)(x_i-a_i)(x_j-a_j)$$

$$= g^{(2)}(0)$$

eg Same $f(x,y) = e^x \cos y$.

Find $P_2(x,y)$ at $a = (0,0)$

using matrix form.

Sol $f(0,0) = 1$

$\nabla f(0,0) = (1, 0)$

$Hf(0,0) = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$

$$P_2(x,y) = f(0,0) + \nabla f(0,0) \begin{bmatrix} x-0 \\ y-0 \end{bmatrix}$$

$$+ \frac{1}{2} [x-0 \quad y-0] \, Hf(0,0) \begin{bmatrix} x-0 \\ y-0 \end{bmatrix}$$

$$= 1 + [1 \quad 0] \begin{bmatrix} x \\ y \end{bmatrix} + \frac{1}{2} [x \quad y] \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

$$= 1 + x + \frac{1}{2} x^2 - \frac{1}{2} y^2$$

eg $g(x,y) = \frac{\ln x}{1-y}$. Find $P_2(x,y)$ at $(1,0)$

Sol $g(1,0) = 0$

$$\nabla g = [g_x, g_y] = \left[\frac{1}{x(1-y)}, \frac{\ln x}{(1-y)^2} \right]$$

$$Hg = \begin{bmatrix} g_{xx} & g_{xy} \\ g_{yx} & g_{yy} \end{bmatrix} = \begin{bmatrix} -\frac{1}{x^2(1-y)} & \frac{1}{x(1-y)^2} \\ \frac{1}{x(1-y)^2} & \frac{2\ln x}{(1-y)^3} \end{bmatrix}$$

$\nabla g(1,0) = [1 \quad 0]$ $\quad Hg(1,0) = \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix}$

$P_2(x,y)$

$$= g(0,0) + \nabla g(0,0) \begin{bmatrix} x-1 \\ y \end{bmatrix} + \frac{1}{2} [x-1 \quad y] \, Hg(1,0) \begin{bmatrix} x-1 \\ y \end{bmatrix}$$

$$= 0 + [1 \quad 0] \begin{bmatrix} x-1 \\ y \end{bmatrix} + \frac{1}{2} [x-1 \quad y] \begin{bmatrix} -1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x-1 \\ y \end{bmatrix}$$

$$= (x-1) - \frac{1}{2}(x-1)^2 + (x-1)y$$

Application to local max / min.

If f is C^2, a is a critical point of f

Then $\nabla f(a) = \vec{0}$. For x near a,

$$f(x) \approx p_2(x)$$
$$= f(a) + \nabla f(a)(x-a) + \frac{1}{2}(x-a)^T Hf(a)(x-a)$$
$$= f(a) + \underbrace{\frac{1}{2}(x-a)^T Hf(a)(x-a)}$$

This term determines whether

$f(x) > f(a)$ or $f(x) < f(a)$

<u>Rmk</u> For $n=1$, i.e. f is 1-variable.

$$\frac{1}{2}(x-a)^T Hf(a)(x-a) = \frac{1}{2} f''(a)(x-a)^2$$

If $f'(a) = 0$, then

$$\begin{cases} f''(a) > 0 \Rightarrow \text{local min at } a \\ f''(a) < 0 \Rightarrow \text{local max at } a \end{cases}$$

(2nd derivative test)

For $n=2$, the 2nd order term is

$$\frac{1}{2}\begin{bmatrix} x-x_0 & y-y_0 \end{bmatrix} \underbrace{\begin{bmatrix} f_{xx}(x_0,y_0) & f_{xy}(x_0,y_0) \\ f_{yx}(x_0,y_0) & f_{yy}(x_0,y_0) \end{bmatrix}} \begin{bmatrix} x-x_0 \\ y-y_0 \end{bmatrix}$$

f is $C^2 \Rightarrow$ Symmetric

To understand nature of critical points, we study <u>quadratic forms</u> of 2 variables.

$$q(x,y) = \begin{bmatrix} x & y \end{bmatrix} \begin{bmatrix} A & B \\ B & C \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$
$$= Ax^2 + 2Bxy + Cy^2$$

Does $q(x,y)$ have a definite sign (always positive or always negative) for $(x,y) \neq (0,0)$?

We can determine it by completing square.

eg 1 $q(x,y) = 2xy$

Note $q(x,y) = \frac{1}{2}(x+y)^2 - \frac{1}{2}(x-y)^2$

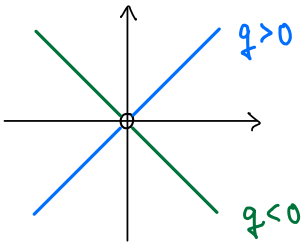
Along $x+y=0$, i.e. $y=-x$,

$$q(x,-x) = -2x^2 < 0 \quad \text{for } x \neq 0$$

Along $x-y=0$, i.e. $y=x$

$$q(x,x) = 2x^2 > 0 \quad \text{for } x \neq 0$$

$\therefore$ q has no definite sign, i.e. indefinite



Clearly $(0,0)$ is a critical point of $q(x,y)$ but neither local max nor min. Such a critical point is called a <u>saddle point</u>.

eg 2 $q(x,y) = 17x^2 - 12xy + 8y^2$. Definite sign?

<u>Soln</u>

$$q(x,y) = 17\left[x^2 - \frac{2\cdot 6}{17}xy + \left(\frac{6}{17}\right)^2 y^2\right] + \left(8 - \frac{36}{17}\right)y^2$$

$$= 17\left(x - \frac{6}{17}y\right)^2 + 10y^2 \quad (*)$$

$\therefore$ $q(x,y) > 0 = q(0,0)$ for $(x,y) \neq (0,0)$

$\therefore$ The critical point $(0,0)$ is a local min.

also global min of $q(x,y)$

<u>Rmk</u> Expression like $(*)$ is called diagonalization of quadratic form. It is not unique! For example $q(x,y) = 5\left(\frac{x+2y}{\sqrt{5}}\right)^2 + 20\left(\frac{2x-y}{\sqrt{5}}\right)^2$ is another diagonalization.

"Orthogonal" change of coordinates

Higher dimension example

eg 3 $q(x,y,z) = xy + yz + zx$

Definite sign for $(x,y,z) \neq (0,0,0)$?

Sol $q = \frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2 + z(x+y)$

Let $u = \frac{x+y}{2}$ $v = \frac{x-y}{2}$. Then

$$
\begin{aligned}
q &= u^2 - v^2 + 2uz \\
&= (u^2 + 2uz + z^2) - v^2 - z^2 \\
&= (u+z)^2 - v^2 - z^2 \\
&= \left(\frac{x+y}{2} + z\right)^2 - \left(\frac{x-y}{2}\right)^2 - z^2 \\
&= \frac{1}{4}(x+y+2z)^2 - \frac{1}{4}(x-y)^2 - z^2
\end{aligned}
$$

positive

negative

On the plane $x+y+2z=0$, i.e. $z = -\frac{x+y}{2}$

$$
\begin{aligned}
q &= q\left(x, y, -\frac{x+y}{2}\right) \\
&= -\frac{1}{4}(x-y)^2 - \frac{1}{4}(x+y)^2 < 0 \quad \text{for } (x,y,z) \neq (0,0,0)
\end{aligned}
$$

Along the line $x-y=z=0$, i.e $y=x$, $z=0$

$$
\begin{aligned}
q(x,y,z) &= q(x,x,0) \\
&= x^2 > 0 \quad \text{for } x \neq 0
\end{aligned}
$$

$\therefore$ The critical point $(0,0,0)$ is a saddle point.

For general theory, need linear algebra:

Diagonalization of quadratic form, eigenvalues ...

Defn Let A be a $n \times n$ symmetric matrix.

Then A is said to be

① **positive definite** if $x^T A x > 0$ for all column vectors $x \in \mathbb{R}^n \setminus \{\vec{0}\}$

② **negative definite** if $x^T A x < 0$ for all column vectors $x \in \mathbb{R}^n \setminus \{\vec{0}\}$

③ **indefinite** if $\exists$ column vectors $x, y \in \mathbb{R}^n \setminus \{\vec{0}\}$ such that $x^T A x > 0$ and $y^T A y < 0$

Rmk These are not all the possible cases:

There are symmetric matrix which is not positive definite, negative definite nor indefinite.

eg

① $$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x^2 + 4y^2 > 0 \quad \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \setminus \{\vec{0}\}$$

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 4 \end{bmatrix}$ is positive definite

② $$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -x^2 - 4y^2 < 0 \quad \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \setminus \{\vec{0}\}$$

$\therefore \begin{bmatrix} -1 & 0 \\ 0 & -4 \end{bmatrix}$ is negative definite

③ $$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = -x^2 + 4y^2$$

$$\begin{bmatrix} 1 & 0 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 1 \\ 0 \end{bmatrix} = -1 < 0$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 4 > 0$$

$\therefore \begin{bmatrix} -1 & 0 \\ 0 & 4 \end{bmatrix}$ is indefinite

④ $$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} = x^2 \geqslant 0 \quad \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$$

$$\begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \end{bmatrix} = 0 \Rightarrow \text{not positive definite}$$

$\therefore \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is neither positive/negative definite nor indefinite

⑤ $$\begin{bmatrix} x & y \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}$$

$$= x^2 + 4xy + 5y^2$$

$$= (x^2 + 4xy + 4y^2) + y^2$$

$$= (x + 2y)^2 + y^2 > 0 \quad \forall \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2 \setminus \{\vec{0}\}$$

$\therefore \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix}$ is positive definite.